

CONTAINMENT RESULTS FOR IDEALS OF VARIOUS CONFIGURATIONS OF POINTS IN \mathbf{P}^N

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ABSTRACT. Guided by evidence coming from a few key examples and attempting to unify previous work of Chudnovsky, Esnault-Viehweg, Eisenbud-Mazur, Ein-Lazarsfeld-Smith, Hochster-Huneke and Bocci-Harbourne, Harbourne and Huneke recently formulated a series of conjectures that relate symbolic and regular powers of ideals of fat points in \mathbf{P}^N . In this paper we propose another conjecture along the same lines (Conjecture 3.9), and we verify it and the conjectures of Harbourne and Huneke for a variety of configurations of points.

1. INTRODUCTION

1.1. Historical Overview. The difference between ordinary and symbolic powers of ideals underlies many fundamental problems in algebraic geometry and commutative algebra. A manifestation of these differences in algebraic geometry is the occurrence of non-linearly normal embeddings of varieties V in projective space, or, more generally, the typical lack of surjectivity of canonical maps of the form $H^0(V, \mathcal{F})^{\otimes r} \rightarrow H^0(V, \mathcal{F}^{\otimes r})$, given a sheaf of modules \mathcal{F} on V . In commutative algebra these differences are related to the occurrence of associated primes for powers I^r of an ideal which are not associated primes for I itself.

One of the simplest contexts of interest in this regard is that of ideals of points in projective space. So consider the ideal I of a finite set of points $P_1, \dots, P_n \in \mathbf{P}^N$. Thus I is the radical ideal $I = \cap_j I(P_j)$ in the polynomial ring $K[\mathbf{P}^N] = K[x_0, \dots, x_N]$ over the ground field K , where $I(P_j)$ is the ideal generated by all homogeneous polynomials (i.e., by all forms) which vanish at each point P_j . The symbolic power $I^{(m)}$ of I in this case is $\cap_j (I(P_j)^m)$. More generally, if $I \subset K[\mathbf{P}^N]$ is any homogeneous ideal, then the associated primes P_i for I are homogeneous and we have a primary decomposition $I = \cap_i Q_i$ where each Q_i is homogeneous and P_i -primary. Let P'_j be the associated primes for I^m and let $I^m = \cap_j Q'_j$ be a primary decomposition such that each Q'_j is homogeneous and P'_j -primary. Then the m -th symbolic power is $I^{(m)} = \cap_{j: P'_j \subseteq P_i \text{ for some } i} Q'_j$.

For simplicity, we will assume K is algebraically closed. In some cases we will assume K has characteristic 0, but only where we say this explicitly.

Given a homogeneous ideal $(0) \neq J \subseteq K[\mathbf{P}^N]$, $\alpha(J)$ denotes the least degree of a non-zero form in J . It is easy to see that $\alpha(J^r) = r\alpha(J)$, but the behavior of $\alpha(J^{(r)})$ is much more complicated and not well-understood. For an ideal I of a finite set of points of \mathbf{P}^N with $K = \mathbb{C}$, Skoda [Sk], in work on complex functions with applications to number theory, sharpened a result of Waldschmidt [W] by showing $\alpha(I^{(m)})/m \geq \alpha(I)/N$ for all $m \geq 1$. A further refinement, $\alpha(I^{(m)})/m \geq \alpha(I^{(n)})/(n+N-1)$, is given in [W2, Lemme 7.5.2]. Chudnovsky [Ch] improved the original Waldschmidt-Skoda bound

Date: February 15, 2012.

2000 Mathematics Subject Classification. Primary: 13F20, 14C20; Secondary: 13A02, 13C05, 14N05.

Key words and phrases. evolutions, symbolic powers, fat points, homogeneous ideals, polynomial rings, projective space.

The first author was partially supported by GNSAGA of INdAM (Italy).

The third author's work on this project was sponsored by the National Security Agency under Grant/Cooperative agreement "Advances on Fat Points and Symbolic Powers," Number H98230-11-1-0139. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notice.

when $N = 2$ by showing $\alpha(I^{(m)})/m \geq (\alpha(I) + 1)/2$ for all $m \geq 1$ (over any field) and conjectured for any N that $\alpha(I^{(m)})/m \geq (\alpha(I) + N - 1)/N$. Esnault-Viehweg [EV] (using methods of complex algebraic geometry such as vanishing theorems) made partial progress towards these conjectures by showing $\alpha(I^{(m)})/m \geq (\alpha(I^{(n)}) + 1)/(n + N - 1)$ for $m, n \geq 1$.

On a different but actually closely related tack, Ein-Lazarsfeld-Smith [ELS] (using multiplier ideals) and Hochster-Huneke [HoHu] (using tight closure) showed that $I^{(rN)} \subseteq I^r$ for all $r > 0$ (as one case of more general results). This raises the question of what the smallest constant c is such that $I^{(m)} \subseteq I^r$ whenever $m > cr$. The first and third authors [BH2] showed in fact that $c = N$ is optimal (in the sense that for each $c < N$, there is an ideal I of points in \mathbf{P}^N such that $I^{(m)} \subseteq I^r$ fails for some m and r with $m > cr$). The third author (see [B. et al]), following up on questions of Huneke and with the goal of obtaining tighter containments, showed for some ideals I that $I^{(Nr - (N-1))} \subseteq I^r$ holds for all $r > 0$, and conjectured that this holds for all I . Motivated by this, by the third author's observation that $I^{(rN)} \subseteq I^r$ implies Skoda's bound in a characteristic free way (see [HaHu]; also see the discussion in [Sc]—the latter paper also obtains Skoda's result for positive characteristics, using methods growing out of tight closure) and by the Eisenbud-Mazur [EM] conjecture on evolutions, the third author and Huneke [HaHu] formulated additional conjectures, refining previous conjectures and tightening them further by considering containments of $I^{(m)}$ in products of the form $M^j I^r$, where $M \subset K[\mathbf{P}^N]$ is the ideal generated by the variables.

Other than theoretical considerations, these new conjectures are based on only a few key examples. The goal of the present paper is to collect together what is known, and to broaden the base of support of these conjectures by proving additional cases of the conjectures. We also propose a new conjecture, Conjecture 3.9, along the same lines as those of [HaHu], and we verify this new conjecture in a range of cases.

1.2. Technical Overview. Although questions of containments of symbolic powers in ordinary powers is of interest in general (and there is some evidence that the conjectures of [HaHu] hold more generally, not just for ideals of points), symbolic powers of ideals of *fat points* are of special interest, both for their conceptual simplicity and as a starting point for trying to understand these containment problems. To recall, given a finite set of distinct points P_1, \dots, P_n in $K[\mathbf{P}^N]$ and non-negative integers m_1, \dots, m_n , a fat point subscheme is the subscheme defined by an ideal of the form $I = I(P_1)^{m_1} \cap I(P_2)^{m_2} \cap \dots \cap I(P_n)^{m_n}$, where $I(P_i)$ is the ideal generated by the forms that vanish at P_i . The m th symbolic power of such an ideal turns out to be $I^{(m)} = I(P_1)^{mm_1} \cap I(P_2)^{mm_2} \cap \dots \cap I(P_n)^{mm_n}$.

In the paper [HaHu], Harbourne and Huneke consider the following general questions (among others):

Question 1.1. [HaHu, Questions 1.3, 1.4 and Conjecture 4.1.5] Let $R = K[\mathbf{P}^N]$ and $M = (x_0, \dots, x_N)$ be the maximal homogeneous ideal of R . Let $I \subseteq R$ be a homogeneous ideal.

- (1) For which m, i and j do we have $I^{(m)} \subseteq M^j I^i$?
- (2) For which j does $I^{(rN)} \subseteq M^j I^r$ hold for all homogeneous ideals $I \subseteq R$ and all r ?
- (3) For which j does $I^{(rN - N + 1)} \subseteq M^j I^r$ hold, given that $I^{(rN - N + 1)} \subseteq I^r$ holds for all r ?

The first question is a natural outgrowth of the Eisenbud-Mazur conjecture (which concerns containment of symbolic squares $P^{(2)}$ of prime ideals P in MP). Given that it is known that $I^{(rN)} \subseteq I^r$ holds for all $r > 0$, the second question arises naturally if one tries to decrease the gap between $I^{(rN)}$ and I^r . Another way to decrease the gap is by making the exponent of the symbolic power smaller. As discussed above, one cannot in general do this by making the coefficient N of r smaller. This suggested subtracting something off, which led to the conjecture that $I^{(rN - (N-1))} \subseteq I^r$ [B. et al, Conjecture 8.4.2], and given this conjecture it is natural to ask if one can decrease the gap further. This leads to the third question.

In the spirit of Question 1.1, Harbourne and Huneke state a series of conjectures (see Section 3) involving containment of symbolic powers of ideals in their regular powers, as well as bounding the initial degrees of symbolic powers in terms of the initial degrees of the ideal itself. We consider these conjectures specialized to various configurations of points. In Section 2 we recall some facts and prove a few others that will be useful later on. In Section 3 we state the conjectures of interest. In Section 4 we verify that the conjectures hold under the assumption that the symbolic and ordinary powers are the same, such as when the points comprise a complete intersection. In Section 5 we consider the case of points on smooth plane conics. In Section 6 we study certain important special point sets of \mathbf{P}^N called star configurations. In Section 7, we look at sets of points contained in a hyperplane, as a corollary of which we recover and extend a result of [Du]. In Section 8 we investigate general sets of points in the plane. Finally, we conclude in Section 9 with a few additional characteristic 0 results for \mathbf{P}^N .

2. PRELIMINARIES

Given a homogeneous ideal $0 \neq I \subseteq R = K[\mathbf{P}^N]$, let $\alpha(I)$ be the least degree t such that the homogeneous component I_t in degree t is not zero. Thus α is, so to speak, the degree in which the ideal begins. It is also the degree of a generator of least degree, and it is the M -adic order of I (i.e., the largest t such that $I \subseteq M^t$), where M is the maximal homogeneous ideal.

The Hilbert function of I is the function $h_I(t) = \dim_K I_t$ for $t \geq 0$. For $t \gg 0$, h_I is a polynomial. Let $\tau(I)$ be the least degree t such that the Hilbert function becomes equal to the Hilbert polynomial of I , let $\sigma(I) = \tau(I) + 1$ and let $0 \rightarrow F_N \rightarrow \cdots \rightarrow F_0 \rightarrow I \rightarrow 0$ be a minimal free resolution of I over R , where F_i as a graded R -module is $\oplus R[-b_{ij}]$. Then the *Castelnuovo-Mumford regularity* $\text{reg}(I)$ of I is the maximum over all i and j of $b_{ij} - i$.

We say that I is saturated if M is not an associated prime of I . The saturation $\text{sat}(I)$ of I is the smallest homogeneous ideal containing I which is saturated. The saturation degree $\text{satdeg}(I)$ of I is the least degree s such that $(\text{sat}(I))_t = I_t$ for all $t \geq s$. If I defines a 0-dimensional subscheme of \mathbf{P}^N , then $\text{reg}(I)$ is the maximum of $\text{satdeg}(I)$ and $\sigma(\text{sat}(I))$, and so we always have $\text{reg}(I) \geq \text{satdeg}(I)$ and see that $\text{reg}(I) = \sigma(I)$ in the case that I is saturated (see [GGP]).

We can now recall a Postulational Containment Criterion that will be useful in this paper:

Lemma 2.1 (Postulational Criterion 2, [BH2]). *Let $I \subseteq K[\mathbf{P}^N]$ be a homogeneous ideal (not necessarily saturated) defining a 0-dimensional subscheme. If $r \text{reg}(I) \leq \alpha(I^{(m)})$, then $I^{(m)} \subseteq I^r$.*

We also recall one of the main results of [HoHu]. The containment $I^{(Nt)} \subseteq I^t$ is the special case for which $m = 1$.

Theorem 2.2. *Let $I \subseteq K[\mathbf{P}^N]$ be a homogeneous ideal. Then $I^{(t(m+N-1))} \subseteq (I^{(m)})^t$ holds for all $m, t \geq 1$.*

Another useful fact is:

Proposition 2.3. *Let $M \subset K[\mathbf{P}^2]$ be the maximal proper homogeneous ideal (i.e., the irrelevant ideal). If $I \subseteq K[\mathbf{P}^2]$ is a homogeneous ideal such that $I^{(j+1)} \subseteq MI^{(j)}$ and $I^{(2j)} = (I^{(2)})^j$ for all $j \geq 1$, then $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$ holds for all $t \geq 1$ and all $m \geq 1$ and $I^{(t(m+1)-1)} \subseteq M^{t-1}(I^{(m)})^t$ holds for all $t \geq 1$ and all even $m \geq 2$. If, moreover, $I^{(2j+1)} = (I^{(2)})^j I$ holds for all $j \geq 0$, then $I^{(t(m+1)-1)} \subseteq M^{t-1}(I^{(m)})^t$ holds for all $t \geq 1$ and all odd $m \geq 1$.*

Proof. Since $I^{(j+1)} \subseteq MI^{(j)}$, we have $I^{(j+i)} \subseteq M^i I^{(j)}$ for all $i \geq 1$ and all $j \geq 1$. And if $j = ab$, then $I^{(2j)} = (I^{(2)})^j = (I^{(2)})^{ab} = ((I^{(2)})^a)^b = (I^{(2a)})^b$, so whenever m is even we have $I^{(tm)} = (I^{(m)})^t$.

First assume m is even. Then $I^{(tm)} = (I^{(m)})^t$ and hence $I^{(t(m+1))} \subseteq M^t I^{(tm)} = M^t (I^{(m)})^t$, and also $I^{(t(m+1)-1)} = I^{(tm+t-1)} \subseteq M^{t-1} I^{(tm)} = M^{t-1} (I^{(m)})^t$.

Now assume m is odd. Then $I^{(t(m+1))} = (I^{(m+1)})^t$. But $I^{(m+1)} \subseteq MI^{(m)}$, so $I^{(t(m+1))} = (I^{(m+1)})^t \subseteq (MI^{(m)})^t = M^t(I^{(m)})^t$. Finally assume in addition that $I^{(2j+1)} = (I^{(2)})^j I$ holds for $j \geq 0$ and write $m = 2\mu + 1$. If $t = 2\tau$ is even, then we have

$$I^{(t(m+1)-1)} = I^{(4\mu\tau+4\tau-1)} = I^{(4\mu\tau+4(\tau-1)+2+1)} = (I^{(2)})^{2\mu\tau+2(\tau-1)+1} I = I^{(2t\mu)} I^{(2(t-1))} I.$$

But $I^{(2(t-1))} \subseteq M^{t-1} I^{t-1}$ (applying the case already proved with $m = 1$), so we have $I^{(t(m+1)-1)} \subseteq I^{(2t\mu)} M^{t-1} I^{t-1} I = M^{t-1} I^t I^{(2t\mu)} = M^{t-1} (I^{(2\mu)} I)^t = M^{t-1} (I^{(m)})^t$. If, on the other hand, $t = 2\tau + 1$ is odd, then $I^{(t(m+1)-1)} = (I^{(2)})^{2\mu\tau+2\tau+\mu} I = (I^{(2\mu)})^t I^{(2(t-1))} I \subseteq (I^{(2\mu)})^t M^{t-1} I^{t-1} I = M^{t-1} (I^{(m)})^t$. \square

The next result is a special case of Proposition 4.2.3 of [Bo].

Lemma 2.4. *Assume K has characteristic 0. Let $I \subseteq K[\mathbf{P}^N] = K[x_0, \dots, x_N]$ be the radical ideal of a finite set of points P_1, \dots, P_n and let $M \subset K[\mathbf{P}^N]$ be the maximal proper homogeneous ideal (i.e., the irrelevant ideal). Then $I^{(j+1)} \subseteq MI^{(j)}$ for each $j \geq 1$.*

Proof. Let $F \in I^{(j+1)}$ be homogeneous. Since $F \in (I(P_i))^{j+1}$ for each i , fixing i and taking coordinates x_0, \dots, x_N such that x_ℓ vanishes at P_i for $\ell > 0$, F is a sum of monomials in x_0, \dots, x_N of degree at least $j+1$ in the variables x_1, \dots, x_N . Thus for each $0 \leq \ell \leq N$, the degree in the variables x_1, \dots, x_N of each term of $\partial F / \partial x_\ell$ is at least j , hence $\partial F / \partial x_\ell \in (I(P_i))^j$. Since the partials with respect to one set of coordinates are linear combinations of the partials with respect to any other linear change of coordinates, we see for any choice of coordinates x_0, \dots, x_N on \mathbf{P}^N that $\partial F / \partial x_\ell \in I(P_i)^j$ for each ℓ and i , hence $\partial F / \partial x_\ell \in I^{(j)}$ for each ℓ .

Applying Euler's identity for a homogeneous polynomial G of positive degree (that $\deg(G)G = \sum_\ell x_\ell \partial G / \partial x_\ell$) we see F is contained in $MI^{(j)}$. \square

This raises the following question:

Question 2.5. Let $I \subsetneq K[\mathbf{P}^N]$ be any proper homogeneous ideal where K has arbitrary characteristic, and let $M \subset K[\mathbf{P}^N]$ be the maximal proper homogeneous ideal (i.e., the irrelevant ideal). Is it true that $I^{(j+1)} \subseteq MI^{(j)}$ for each $j \geq 1$?

The following result can be useful in some cases; it is a variation of [HaHu, Proposition 2.3].

Lemma 2.6. *Let $I \subset K[\mathbf{P}^N]$ be a homogeneous ideal defining a zero dimensional subscheme of \mathbf{P}^N and let $M \subset K[\mathbf{P}^N]$ be the irrelevant ideal (i.e., the ideal generated by the indeterminates of $K[\mathbf{P}^N]$).*

- (a) *If $\alpha(I^{(t(m+N-1))}) \geq t \operatorname{reg}(I^{(m)}) + t(N-1)$, then $I^{(t(m+N-1))} \subseteq M^{t(N-1)}(I^{(m)})^t$.*
- (b) *If $I^{(t(m+N-1)-(N-1))} \subseteq (I^{(m)})^t$ and $\alpha(I^{(t(m+N-1)-(N-1))}) \geq t \operatorname{reg}(I^{(m)}) + (t-1)(N-1)$, then*

$$I^{(t(m+N-1)-(N-1))} \subseteq M^{(t-1)(N-1)}(I^{(m)})^t.$$

Proof. We know, by [GGP, Theorem 1.1], that $\operatorname{reg}((I^{(m)})^t) \leq t \operatorname{reg}(I^{(m)})$. In particular, $(I^{(m)})^t$ is generated in degree at most $t \operatorname{reg}(I^{(m)})$. Thus for any degree $s \geq t \operatorname{reg}(I^{(m)})$ we have $M_1((I^{(m)})^t)_s = ((I^{(m)})^t)_{s+1}$ and hence $(M^i)_i((I^{(m)})^t)_s = ((I^{(m)})^t)_{s+i}$. But $((M^i)_i(I^{(m)})^t)_s \subseteq (M^i(I^{(m)})^t)_{s+i} \subseteq ((I^{(m)})^t)_{s+i}$ so $(M^i(I^{(m)})^t)_{s+i} = ((I^{(m)})^t)_{s+i}$ if $s \geq t \operatorname{reg}(I^{(m)})$.

(a) By Theorem 2.2 we have $I^{(t(m+N-1))} \subseteq (I^{(m)})^t$. Thus we have $(I^{(t(m+N-1))})_s \subseteq ((I^{(m)})^t)_s = (M^{t(N-1)}(I^{(m)})^t)_s$ for $s \geq t \operatorname{reg}(I^{(m)}) + t(N-1)$. But $(I^{(t(m+N-1))})_s = 0$ for $s < \alpha(I^{(t(m+N-1))})$, so if $\alpha(I^{(t(m+N-1))}) \geq t \operatorname{reg}(I^{(m)}) + t(N-1)$, then we have $(I^{(t(m+N-1))})_s \subseteq ((I^{(m)})^t)_s$ for all $s \geq 0$, which implies the result.

(b) By assumption we have $I^{(t(m+N-1)-(N-1))} \subseteq (I^{(m)})^t$. Now mimic the proof of (a). We have $(I^{(t(m+N-1)-(N-1))})_s \subseteq ((I^{(m)})^t)_s = (M^{(t-1)(N-1)}(I^{(m)})^t)_s$ for $s \geq t \operatorname{reg}(I^{(m)}) + (t-1)(N-1)$. But $(I^{(t(m+N-1)-(N-1))})_s = 0$ for $s < \alpha(I^{(t(m+N-1)-(N-1))})$, so if $\alpha(I^{(t(m+N-1)-(N-1))}) \geq t \operatorname{reg}(I^{(m)}) +$

$(t-1)(N-1)$, then we have $(I^{(t(m+N-1)-(N-1))})_s \subseteq ((I^{(m)})^t)_s$ for all $s \geq 0$, which implies the result. \square

3. THE CONJECTURES

For the reader's convenience, we list here the conjectures we will be considering.

Conjecture 3.1 ([HaHu, Conjecture 2.1]). *Let $I = \cap_{i=1}^n I(P_i)^{m_i} \subset K[\mathbf{P}^N]$ be any fat points ideal and let $M \subset K[\mathbf{P}^N]$ be the maximal proper homogeneous ideal (i.e., the irrelevant ideal). Then $I^{(rN)} \subseteq M^{r(N-1)} I^r$ holds for all $r > 0$.*

Conjecture 3.2 ([B. et al, Conjecture 8.20]). *Let $I \subseteq K[\mathbf{P}^N]$ be a homogeneous ideal. Then $I^{(rN-(N-1))} \subseteq I^r$ holds for all r .*

Conjecture 3.3 ([HaHu, Conjecture 4.1.4]). *Let $I \subseteq K[\mathbf{P}^2]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^2$. Then $I^{(m)} \subseteq I^r$ holds whenever $m/r \geq 2\alpha(I)/(\alpha(I) + 1)$.*

Conjecture 3.4 ([HaHu, Conjecture 4.1.5]). *Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$ and let $M \subset K[\mathbf{P}^N]$ be the maximal proper homogeneous ideal (i.e., the irrelevant ideal). Then $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$ holds for all $r \geq 1$.*

Conjecture 3.5 ([HaHu, Conjecture 4.1.8]). *Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then*

$$\alpha(I^{(rN-(N-1))}) \geq r\alpha(I) + (r-1)(N-1)$$

for every $r > 0$.

Conjecture 3.6 ([HaHu, Question 4.2.1]). *Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then*

$$\frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \leq \frac{\alpha(I^{(r)})}{r}$$

for all $r > 0$.

Conjecture 3.7 ([HaHu, Question 4.2.2]). *Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$ for $N \geq 2$ and let $M \subset K[\mathbf{P}^N]$ be the maximal proper homogeneous ideal (i.e., the irrelevant ideal). Then $I^{(t(m+N-1))} \subseteq M^t (I^{(m)})^t$.*

Conjecture 3.8 ([HaHu, Question 4.2.3]). *Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then $I^{(t(m+N-1))} \subseteq M^{t(N-1)} (I^{(m)})^t$ where M is the irrelevant ideal.*

Note that if we take $m = 1$ in Conjecture 3.8 we recover Conjecture 3.1. This suggests the following conjecture, which in the same way implies Conjectures 3.2 and 3.4. It also completes a pair of analogies: Conjecture 3.1 is to Conjecture 3.4 as Conjecture 3.8 is to the second part of the following conjecture, and Conjecture 3.2 is to Conjecture 3.4 as the first part of the following conjecture is to the second part.

Conjecture 3.9. *Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then $I^{(t(m+N-1)-N+1)} \subseteq (I^{(m)})^t$ and $I^{(t(m+N-1)-N+1)} \subseteq M^{(t-1)(N-1)} (I^{(m)})^t$ hold for all $m \geq 1$, where M is the irrelevant ideal.*

As we just saw, some conjectures are stronger than others. In fact the following implications hold:

Proposition 3.10.

- (1) Conjecture 3.4 implies Conjecture 3.5.
- (2) Conjecture 3.8 implies Conjectures 3.1, 3.6 and 3.7.

(3) Conjecture 3.9 implies Conjectures 3.2, 3.4 and 3.5.

Proof. (1) Suppose Conjecture 3.4 holds. From $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$ one has

$$\alpha(I^{(rN-(N-1))}) \geq \alpha(M^{(r-1)(N-1)} I^r) = \alpha(M^{(r-1)(N-1)}) + \alpha(I^r) = (r-1)(N-1) + r\alpha(I)$$

and hence Conjecture 3.5 holds.

(2) Suppose Conjecture 3.8 holds. The implication of Conjecture 3.7 is obvious. By Conjecture 3.8, taking $t = r$ and $m = 1$, one has $I^{(rN)} \subseteq M^{r(N-1)}(I^{(m)})^r = M^{r(N-1)} I^r$. Hence Conjecture 3.1 holds. Again from $I^{(t(m+N-1))} \subseteq M^{t(N-1)}(I^{(m)})^t$ of Conjecture 3.8, and from $(I^{(t)})^{m+N-1} \subseteq I^{(t(m+N-1))}$ one has

$$t(\alpha(I^{(m)}) + N - 1) = \alpha(M^{t(N-1)}(I^{(m)})^t) \leq \alpha((I^{(t)})^{m+N-1}) = (m + N - 1)\alpha(I^{(t)})$$

from which Conjecture 3.6 follows.

(3) Conjecture 3.9 implies Conjectures 3.2 and 3.4 by taking $m = 1$; then note that Conjecture 3.4 implies Conjecture 3.5 by (1). \square

4. ASSUMING SYMBOLIC EQUALS ORDINARY

Suppose $I \subsetneq K[\mathbf{P}^N]$ is a homogeneous ideal such that $I^{(r)} = I^r$ for all r , such as is the case if I is a complete intersection (see [ZS, Lemma 5, Appendix 6]). Note that $I \subseteq M$.

Conjecture 3.1, $I^{(rN)} \subseteq M^{r(N-1)} I^r$, holds since $I^{(rN)} = I^{rN} = I^{r(N-1)} I^r \subseteq M^{r(N-1)} I^r$.

Conjecture 3.2, $I^{(rN-(N-1))} \subseteq I^r$, holds since $I^{(rN-(N-1))} = I^{rN-N+1}$ and since $rN - N + 1 \geq r$, given $N \geq 1$, implies $I^{rN-N+1} \subseteq I^r$.

Conjecture 3.3, that $m/r \geq 2\alpha(I)/(\alpha(I) + 1)$ implies $I^{(m)} \subseteq I^r$ for $N = 2$, holds since $I^{(m)} = I^m$ and since $2\alpha(I)/(\alpha(I) + 1) \geq 1$, so $m/r \geq 2\alpha(I)/(\alpha(I) + 1)$ implies $m \geq r$, hence $I^m \subseteq I^r$.

Conjecture 3.4, $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$, holds since $I^{(rN-(N-1))} = I^{rN-N+1} = I^{(r-1)(N-1)} I^r \subseteq M^{(r-1)(N-1)} I^r$.

Conjecture 3.5, $\alpha(I^{(rN-(N-1))}) \geq r\alpha(I) + (r-1)(N-1)$, holds since $\alpha(I) \geq 1$ but $\alpha(I^{(rN-(N-1))}) = \alpha(I^{rN-N+1}) = (rN - N + 1)\alpha(I) \geq r\alpha(I) + (r-1)(N-1)$ is equivalent to $\alpha(I) \geq 1$.

Conjecture 3.6, $\frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \leq \frac{\alpha(I^{(r)})}{r}$, holds since $\alpha(I) \geq 1$ but

$$\frac{m\alpha(I) + N - 1}{m + N - 1} = \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \leq \frac{\alpha(I^{(r)})}{r} = \frac{\alpha(I^r)}{r} = \frac{r\alpha(I)}{r} = \alpha(I)$$

is equivalent to $m\alpha(I) + N - 1 \leq \alpha(I)(m + N - 1)$ and hence to $1 \leq \alpha(I)$.

Conjecture 3.7, that $I^{(t(m+N-1))} \subseteq M^t(I^{(m)})^t$ holds for $N \geq 2$, is true since $I^{(t(m+N-1))} = I^{t(m+N-1)} = I^t I^{t(m+N-2)} \subseteq M^t I^{mt} = M^t(I^{(m)})^t$.

Conjecture 3.8, $I^{(t(m+N-1))} \subseteq M^{t(N-1)}(I^{(m)})^t$, holds since $I^{(t(m+N-1))} = I^{t(m+N-1)} = I^{t(N-1)} I^{tm} \subseteq M^{t(N-1)} I^{mt} = M^{t(N-1)}(I^{(m)})^t$.

As for Conjecture 3.9, the second part, $I^{(t(m+N-1)-N+1)} \subseteq M^{(t-1)(N-1)}(I^{(m)})^t$, holds (and hence so does the first part $I^{(t(m+N-1)-N+1)} \subseteq (I^{(m)})^t$), since

$$I^{(t(m+N-1)-N+1)} = I^{t(m+N-1)-N+1} = I^{(t-1)(N-1)} I^{mt} \subseteq M^{(t-1)(N-1)} I^{mt} = M^{(t-1)(N-1)}(I^{(m)})^t.$$

In particular, if $N = 2$, all of the conjectures hold if I is the radical ideal of a single point, or the radical ideal of n points on a smooth plane conic when n is even, since then the points comprise a complete intersection. In the next section we consider the case of the radical ideal of an odd number of points on a smooth plane conic. (One can also ask about the case of ideals of points on reducible conics; i.e., of points on a pair of lines in the plane. See [DJ] for some results in this direction.)

5. ODD NUMBERS OF POINTS ON A SMOOTH PLANE CONIC

The case of $n = 3$ points on a smooth plane conic C (so $N = 2$) is somewhat special and mostly known, so we treat that case with some initial remarks. Conjecture 3.1 holds by [HaHu, Corollary 3.9] since $n = 3$ points on a smooth conic comprise a star configuration. Conjectures 3.2 and 3.5 follow from Conjecture 3.4, which holds for the $n = 3$ case by [HaHu, Corollary 4.1.7]. Conjecture 3.3 holds for $n = 3$ since $\alpha(I) = 2$ so $m/r \geq 2\alpha(I)/(\alpha(I) + 1)$ is equivalent to $3m/2 \geq 2r$, but $\text{reg}(I) = 2$ and, by [BH2, Lemma 2.4.1], $3m/2 \leq \alpha(I^{(m)})$, so assuming $3m/2 \geq 2r$ we have $r \text{reg}(I) = 2r \leq 3m/2 \leq \alpha(I^{(m)})$, hence $I^{(m)} \subseteq I^r$ by Lemma 2.1. Conjectures 3.6 and 3.7 follow from Conjecture 3.8, which holds for $n = 3$ in characteristic 0 by Corollary 5.3 below.

Before continuing, we introduce another useful numerical character: for any homogeneous ideal $0 \neq I \subseteq R = K[\mathbf{P}^2]$, let $\beta(I)$ be the smallest integer t such that I_t contains a regular sequence of length two.

Lemma 5.1. *Let I be the radical ideal of $n \geq 5$ points on a smooth conic C in \mathbf{P}^2 , where n is odd. Then $I^{(mr)} = (I^{(m)})^r$ for any even m .*

Proof. We first note that $I^{(2r)} = (I^{(2)})^r$. This is because $\alpha(I^{(2r)}) = 4r$ and $\beta(I^{(2r)}) = rn$ (because it is known [H2] that the only fixed component for the linear system of curves of degree d through $n \geq 5$ points of multiplicity r is the conic C through all n points, and these occur only if forced by Bézout). Now we have $\alpha(I^{(2r)})\beta(I^{(2r)}) = (2r)^2n$, hence $I^{(2r)} = (I^{(2)})^r$ by Proposition 3.5 of [HaHu]. Thus for $m = 2s$ we have $I^{(mr)} = I^{(2sr)} = (I^{(2)})^{sr} = ((I^{(2)})^s)^r = (I^{(m)})^r$. \square

We now deal with the case of odd symbolic powers:

Lemma 5.2. *If I is the radical ideal of $n \geq 5$ points on a smooth conic C in \mathbf{P}^2 with n odd, then $I^{(2r+1)} = (I^{(2)})^r I$.*

Proof. It is enough to show that $I^{(2r+1)} = (I^{(2r)})I$, since $I^{(2r)} = (I^{(2)})^r$. Since $(I^{(2r)})I \subseteq I^{(2r+1)}$, it is enough to show $(I^{(2r+1)})_t \subseteq ((I^{(2r)})I)_t$ for every $t \geq \alpha(I^{(2r+1)}) = 2(2r+1)$. Let $m = 2r+1$ and let F be the form defining C . If $4r+2 \leq t \leq (mn-1)/2$, then $2t-mn < 0$, so by Bézout we know that C is a fixed component of $(I^{(2r+1)})_t$, hence $(I^{(2r+1)})_t = F \cdot (I^{(2r)})_{t-2} \subseteq (I^{(2r)})_{t-2} I_2 \subseteq ((I^{(2r)})I)_t$. So say $t \geq (mn+1)/2$. Note that $I_{(n+1)/2}$ is fixed component free since $2(n+1)/2 - n \geq 0$, and $(I^{(m-1)})_j$ is fixed component free for $j \geq n(m-1)/2$, since $2j - (m-1)n \geq 2n(m-1)/2 - (m-1)n \geq 0$, so by [BH1, Proposition 2.4] we have $(I^{(m-1)})_j I_{(n+1)/2} = (I^{(m)})_{j+(n+1)/2}$. But let $j = t - (n+1)/2$. Then $t \geq (mn+1)/2$ implies $j \geq n(m-1)/2$, so we have $(I^{(m)})_t = (I^{(m-1)})_j I_{(n+1)/2} \subseteq (I^{(m-1)})_j I_t$. \square

We now show that Conjectures 3.1 through 3.6 hold in the case that $N = 2$ for $n \geq 5$ points on a smooth plane conic if n is odd.

Conjecture 3.1, $I^{(rN)} \subseteq M^{r(N-1)}I^r$, holds: Suppose we show $I^{(2)} \subseteq MI$. Then we would have $I^{(2r)} = (I^{(2)})^r \subseteq (MI)^r = M^r I^r$, as required. Thus it's enough now to show that $(I^{(2)})_t \subseteq M_i I_{t-i}$ for some i for each $t \geq \alpha(I^{(2)})$. But for $\alpha(I^{(2)}) \leq t < \beta(I^{(2)}) = n$, we know C is a fixed component of $(I^{(2)})_t$. Since $F \in M_2$ for the form F defining C , we have $(I^{(2)})_t = F(I_{t-2}) \subseteq M_2 I_{t-2}$, as needed. For $t \geq n$, we have $M_1 I_{t-1} = I_t$ by [BH1, Lemma 2.2] since $t \geq n = \beta(I^{(2)}) > \beta(I)$, so $(I^{(2)})_t \subseteq I_t = M_1 I_{t-1}$.

Conjecture 3.2, $I^{(rN-(N-1))} \subseteq I^r$, holds: Here we want to verify that $I^{(2r-1)} \subseteq I^r$, which is $I^{(2m+1)} \subseteq I^{m+1}$ if we take $r = m + 1$. But $I^{(2m+1)} = (I^{(2)})^m I \subseteq (MI)^m I = M^m I^{m+1} \subseteq I^{m+1}$, as required.

Conjecture 3.3, that $m/r \geq 2\alpha(I)/(\alpha(I) + 1)$ implies $I^{(m)} \subseteq I^r$, holds: To see this it is helpful to recall the resurgence, $\rho(J)$ of a homogeneous ideal $(0) \neq J \subsetneq k[\mathbf{P}^N]$, defined to be the supremum of the ratios m/r such that $I^{(m)} \not\subseteq I^r$. In the present situation, it is enough to have $\rho(I) < 2\alpha(I)/(\alpha(I) + 1)$, but by [BH1, Theorem 3.4] we have $\rho(n) = (n + 1)/n$. Since $\alpha(I) = 2$ and $n \geq 5$ we have $\rho(I) = (n + 1)/n < 4/3 = 2\alpha(I)/(\alpha(I) + 1)$.

Conjecture 3.4, $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$, holds: Here we want to verify that $I^{(2r-1)} \subseteq M^{r-1} I^r$, which is $I^{(2m+1)} \subseteq M^m I^{m+1}$ if we take $r = m + 1$. But $I^{(2m+1)} = (I^{(2)})^m I \subseteq (MI)^m I = M^m I^{m+1}$, as required.

Conjecture 3.5, $\alpha(I^{(rN-(N-1))}) \geq r\alpha(I) + (r - 1)(N - 1)$, holds: Here we want to verify that $\alpha(I^{(2r-1)}) \geq r\alpha(I) + r - 1$, but $\alpha(I^{(2r-1)}) = 2(2r - 1) \geq 2r + r - 1 = r\alpha(I) + r - 1$ holds since $r \geq 1$.

Conjecture 3.6, $\frac{\alpha(I^{(m)})+N-1}{m+N-1} \leq \frac{\alpha(I^{(r)})}{r}$, holds: Here we want to verify that $\frac{\alpha(I^{(m)})+1}{m+1} \leq \frac{\alpha(I^{(r)})}{r}$ for all $r \geq 1$, but

$$\frac{\alpha(I^{(m)}) + 1}{m + 1} = \frac{2m + 1}{m + 1} \leq 2 = \frac{2r}{r} = \frac{\alpha(I^{(r)})}{r}$$

is clear.

In case $N = 2$, Conjecture 3.7 and Conjecture 3.8 both assert that $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$, while Conjecture 3.9 asserts $I^{(t(m+1)-1)} \subseteq M^{t-1}(I^{(m)})^t \subseteq (I^{(m)})^t$. We note that the next result holds also for $n = 1$ and $n = 3$, but when $n = 1$ we have a single point and that case was dealt with in Section 4, while for $n = 3$ the points comprise a star configuration, which is dealt with in Section 6.

Corollary 5.3. *Let $N = 2$ and let $n \geq 5$ be odd. Then Conjectures 3.7, 3.8 and 3.9 hold for the radical ideal of any n points on a smooth plane conic if $\text{char}(K) = 0$.*

Proof. Let I be the radical ideal of the points. For $n \geq 5$ odd we have $I^{(2j)} = (I^{(2)})^j$ by Lemma 5.1, and we have $I^{(2r+1)} = (I^{(2)})^r I$ by Lemma 5.2. The result now follows in characteristic 0 from Proposition 2.3 by applying Lemma 2.4. \square

6. STAR CONFIGURATIONS OF POINTS IN PROJECTIVE SPACE

Let I be the radical ideal of the configuration of $\binom{s}{N}$ points given by the pair-wise intersection of $s \geq N$ hyperplanes in \mathbf{P}^N , assuming no $N + 1$ of the hyperplanes meet at a point. As usual, let $M \subset K[\mathbf{P}^N]$ be the maximal proper homogeneous ideal (i.e., the irrelevant ideal).

Conjecture 3.1, that $I^{(rN)} \subseteq M^{r(N-1)} I^r$, holds by [HaHu, Corollary 3.9].

Conjecture 3.2, that $I^{(rN-(N-1))} \subseteq I^r$, holds by [B. et al, Example 8.4.8].

Conjecture 3.3, that $I^{(m)} \subseteq I^r$ if $m/r \geq 2\alpha(I)/(\alpha(I) + 1)$ and $N = 2$, holds (see the discussion after Conjecture 4.1.4 of [HaHu]).

Conjecture 3.4, that $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$, holds by [HaHu, Corollary 4.1.7].

Conjecture 3.5, that $\alpha(I^{(rN-(N-1))}) \geq r\alpha(I) + (r-1)(N-1)$, follows from Conjecture 3.4.

Conjecture 3.6, that $\frac{\alpha(I^{(m)})+N-1}{m+N-1} \leq \frac{\alpha(I^{(r)})}{r}$, and Conjecture 3.7, that $I^{(t(m+N-1))} \subseteq M^t(I^{(m)})^t$ for $N \geq 2$, follow from Conjecture 3.8, that $I^{(t(m+N-1))} \subseteq M^{t(N-1)}(I^{(m)})^t$. We will verify Conjecture 3.8 in case $N = 2$ when K has characteristic 0. For $N = 2$, we have $I^{(2m)} = (I^{(2)})^m$ for all $m \geq 1$ by [HaHu, Corollary 3.9], and we have $I^{(m+1)} \subseteq MI^{(m)}$ by Lemma 2.4. Thus Conjecture 3.8 holds when $N = 2$ and $\text{char}(K) = 0$ by Proposition 2.3. This same proof shows that Conjecture 3.9, that $I^{(t(m+N-1)-1)} \subseteq M^{(t-1)(N-1)}(I^{(m)})^t$, holds in case $N = 2$ when K has characteristic 0 and m is even, since for a star configuration in \mathbf{P}^2 we have $I^{(2j)} = (I^{(2)})^j$ by [HaHu, Corollary 3.9]. It also holds for m odd by the same proof, except to apply Proposition 2.3 we must check that $I^{(2j+1)} = (I^{(2)})^j I$, which we now do. We always have $(I^{(2)})^j I \subseteq I^{(2j+1)}$ so we must check the reverse containment. The case $j = 0$ is clear, so assume $j > 0$. By [BH2, Lemma 2.4.2], $s-1 = \alpha(I) = \text{reg}(I)$ and $\text{reg}(I^m) \leq m \text{reg}(I)$ by [GGP, Theorem 1.1]. Thus we have equality of the homogeneous components of the ideals $(I^{(m)})_t = (I^m)_t$ for every degree $t \geq m(s-1)$. In particular, $(I^{(2j+1)})_t = (I^{2j+1})_t \subseteq ((I^{(2)})^j I)_t$. For $t < m(s-1)$, by Bézout's Theorem, each of the s lines is in the zero locus of each element of $(I^{(2j+1)})_t$; i.e., $(I^{(2j+1)})_t = F(I^{(2(j-1)+1)})_{t-s}$, where the form F defines the s lines. By induction we have $F(I^{(2(j-1)+1)})_{t-s} \subseteq F(I^{(2(j-1))})_{t-s} \subseteq (I^{(2j)})_{t-s} = ((I^{(2)})^j I)_{t-s}$. Thus $(I^{(2j+1)})_t \subseteq ((I^{(2)})^j I)_t$ holds for all t , hence $I^{(2j+1)} \subseteq (I^{(2)})^j I$.

7. POINTS IN A HYPERPLANE

Suppose distinct points $P_1, \dots, P_n \in \mathbf{P}^{N+1}$ lie in a hyperplane H of \mathbf{P}^{N+1} . We can regard H as \mathbf{P}^N . There is now some ambiguity when using the notation $m_1 P_1 + \dots + m_n P_n$, since $m_1 P_1 + \dots + m_n P_n$ could denote a fat point subscheme of \mathbf{P}^N or of \mathbf{P}^{N+1} , and these are not the same. For example, consider the point $P = (0, 0, 0, 1) \in \mathbf{P}^3$. Taking coordinates $K[\mathbf{P}^3] = K[x_0, \dots, x_3]$, $P \in H$ where H is defined as $x_0 = 0$. We can identify H with \mathbf{P}^2 , where $K[\mathbf{P}^2] = K[x_1, \dots, x_3]$. If we use mP to denote the fat point subscheme of \mathbf{P}^3 , we have $I(mP) = (x_0, x_1, x_2)^m$, but, as a fat point subscheme of \mathbf{P}^2 which becomes a subscheme of \mathbf{P}^3 by the inclusion $\mathbf{P}^2 \subset \mathbf{P}^3$, mP has ideal $I(mP) = (x_1, x_2)^m + (x_0)$.

To resolve the ambiguity, given $Z = m_1 P_1 + \dots + m_n P_n$ for points $P_i \in H$, we will use the notation $Z_{\mathbf{P}^N}$ and $Z_{\mathbf{P}^{N+1}}$ to indicate whether we are regarding Z as the fat point subscheme of \mathbf{P}^N or of \mathbf{P}^{N+1} , respectively.

For explicitness (but without loss of generality) we assume $K[\mathbf{P}^{N+1}] = K[x_0, \dots, x_{N+1}]$ and H is defined by $x_0 = 0$, so $K[\mathbf{P}^N] = K[H] = K[x_1, \dots, x_{N+1}]$.

Let $Z = P_1 + \dots + P_n$, let $I = I(Z_{\mathbf{P}^N})$, let $\tilde{I} = IK[\mathbf{P}^{N+1}]$ be the extension and let $\hat{I} = I(Z_{\mathbf{P}^{N+1}})$. Also, let $M = (x_1, \dots, x_{N+1}) \subset K[\mathbf{P}^N]$, let $\tilde{M} = MK[\mathbf{P}^{N+1}]$ and let $\widehat{M} = (x_0, \dots, x_{N+1}) \subset K[\mathbf{P}^{N+1}]$.

Proposition 7.1. *Let $I, \tilde{I}, \hat{I}, M, \tilde{M}$ and \widehat{M} be as above. Moreover, for parts (a) and (c), assume K has characteristic 0.*

- (a) *If Conjecture 3.1 holds for I , then it holds for \hat{I} ; i.e., if $I^{(rN)} \subseteq M^{r(N-1)} I^r$ holds, then so does $\hat{I}^{(r(N+1))} \subseteq \widehat{M}^{rN} \hat{I}^r$.*
- (b) *If Conjecture 3.2 holds for I , then it holds for \hat{I} ; in fact, if $I^{(rN-(N-1))} \subseteq I^r$ holds for all $r \geq 1$, then so does $\hat{I}^{(r(N+1)-N)} \subseteq \hat{I}^{(rN-(N-1))} \subseteq \hat{I}^r$.*
- (c) *If Conjecture 3.4 holds for I , then it holds for \hat{I} ; i.e., if $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$ holds, then so does $\hat{I}^{(r(N+1)-N)} \subseteq \widehat{M}^{(r-1)N} \hat{I}^r$.*

Proof. (a) By [FHL], $\widehat{I}^{(m)} = (x_0^m) + x_0^{m-1}\widetilde{I} + \dots + x_0\widetilde{I}^{(m-1)} + \widetilde{I}^{(m)}$ for any $m \geq 1$. Thus $\widehat{I} = (x_0) + \widetilde{I}$, so $\widehat{M}^{rN}\widehat{I}^r$ is equal to $((x_0) + \widetilde{M})^{rN}((x_0) + \widetilde{I})^r$, which expands to

$$((x_0)^{rN} + x_0^{rN-1}\widetilde{M} + \dots + x_0\widetilde{M}^{rN-1} + \widetilde{M}^{rN})((x_0)^r + x_0^{r-1}\widetilde{I} + \dots + x_0\widetilde{I}^{r-1} + \widetilde{I}^r).$$

Multiplying this out and combining terms with equal powers of x_0 gives

$$\begin{aligned} & (x_0)^{(N+1)r} + \\ & (x_0)^{(N+1)r-1}(\widetilde{M} + \widetilde{I}) + \\ & (x_0)^{(N+1)r-2}(\widetilde{M}^2 + \widetilde{M}\widetilde{I} + \widetilde{I}^2) + \\ & \dots + \\ & (x_0)^{(N+1)r-r}(\widetilde{M}^r + \widetilde{M}^{r-1}\widetilde{I} + \dots + \widetilde{I}^r) + \\ & (x_0)^{rN-1}(\widetilde{M}^{r+1} + \widetilde{M}^r\widetilde{I} + \dots + \widetilde{M}\widetilde{I}^r) + \\ & (x_0)^{rN-2}(\widetilde{M}^{r+2} + \widetilde{M}^{r+1}\widetilde{I} + \dots + \widetilde{M}^2\widetilde{I}^r) + \\ & \dots + \\ & (x_0)^{rN-(rN-r)}(\widetilde{M}^{r+rN-r} + \widetilde{M}^{r+rN-r-1}\widetilde{I} + \dots + \widetilde{M}^{r+rN-2r}\widetilde{I}^r) + \\ & (x_0)^{r-1}(\widetilde{M}^{rN}\widetilde{I} + \widetilde{M}^{rN-1}\widetilde{I}^2 + \dots + \widetilde{M}^{rN-(r-1)}\widetilde{I}^r) + \\ & \dots + \\ & (x_0)(\widetilde{M}^{rN}\widetilde{I}^{r-1} + \widetilde{M}^{rN-1}\widetilde{I}^r) + \\ & \widetilde{M}^{rN}\widetilde{I}^r. \end{aligned}$$

Since $\widetilde{I} \subseteq \widetilde{M}$, the first term in each row of the displayed sum of ideals above contains the other terms in that row; e.g., in row 3 we have

$$\widetilde{I}^2 \subseteq \widetilde{M}\widetilde{I} \subseteq \widetilde{M}^2.$$

Thus we get

$$\widehat{M}^{rN}\widehat{I}^r = (x_0)^{(N+1)r} + (x_0)^{(N+1)r-1}\widetilde{M} + \dots + (x_0)^r\widetilde{M}^{rN} + (x_0)^{r-1}\widetilde{M}^{rN}\widetilde{I} + \dots + x_0\widetilde{M}^{rN}\widetilde{I}^{r-1} + \widetilde{M}^{rN}\widetilde{I}^r.$$

Since

$$\widehat{I}^{(r(N+1))} = (x_0^{(N+1)r}) + x_0^{(N+1)r-1}\widetilde{I} + \dots + x_0\widetilde{I}^{((N+1)r-1)} + \widetilde{I}^{((N+1)r)},$$

to show $\widehat{I}^{(r(N+1))} \subseteq \widehat{M}^{rN}\widehat{I}^r$ it suffices to show that $x_0^{(N+1)r-j}\widetilde{I}^{(j)} \subseteq x_0^{(N+1)r-j}\widetilde{M}^j$ for $j = 0, \dots, Nr$, and that $x_0^{(N+1)r-j}\widetilde{I}^{(j)} \subseteq x_0^{(N+1)r-j}\widetilde{M}^{Nr}\widetilde{I}^{j-Nr}$ for $j = Nr+1, \dots, (N+1)r$.

However, $\widetilde{I} \subseteq \widetilde{M}$, so $\widetilde{I}^t \subseteq \widetilde{M}^t$ for all $t > 0$. Since adding a variable to a polynomial ring gives a flat extension, primary decompositions of ideals in $K[\mathbf{P}^N]$ extend to primary decompositions in $K[\mathbf{P}^{N+1}]$ (see [M], Theorem 13, or Exercise 7, [AM]). Thus saturating with respect to \widetilde{M} gives $\widetilde{I}^{(t)} \subseteq \widetilde{M}^{(t)}$, but \widetilde{M}^t is \widetilde{M} -primary so saturation has no effect; i.e., $\widetilde{M}^{(t)} = \widetilde{M}^t$. Thus $\widetilde{I}^{(t)} \subseteq \widetilde{M}^t$, which shows $x_0^{(N+1)r-j}\widetilde{I}^{(j)} \subseteq x_0^{(N+1)r-j}\widetilde{M}^j$. Now we want to show

$$x_0^{(N+1)r-j}\widetilde{I}^{(j)} \subseteq x_0^{(N+1)r-j}\widetilde{M}^{Nr}\widetilde{I}^{j-Nr}.$$

Given that $j > Nr$, by $j - Nr$ applications of Lemma 2.4, we have $I^{(j)} \subseteq M^{j-Nr}I^{(Nr)}$ and hence

$$x_0^{(N+1)r-j}\widetilde{I}^{(j)} \subseteq x_0^{(N+1)r-j}\widetilde{M}^{j-Nr}\widetilde{I}^{(Nr)}.$$

(Here we use the obvious fact that an extension of a product is the product of the extensions, i.e., $(M^{j-Nr}I^{(Nr)})^\sim = \widetilde{M}^{j-Nr}\widetilde{I}^{(Nr)}$, and the less obvious fact that in this situation the extension of the symbolic power is the symbolic power of the extension; i.e., $\widetilde{I}^{(Nr)} = \widetilde{I}^{(Nr)}$. The latter is true

because of the fact quoted above about the preservation of primary decompositions for the extension $K[\mathbf{P}^N] \subset K[\mathbf{P}^{N+1}]$.) Thus $\widehat{I}^{(Nr)} \subseteq \widetilde{M}^{r(N-1)} \widetilde{I}^r$ since by assumption we have $I^{(Nr)} \subseteq M^{r(N-1)} I^r$, hence we obtain

$$x_0^{(N+1)r-j} \widetilde{I}^{(j)} \subseteq x_0^{(N+1)r-j} \widetilde{M}^{j-Nr} \widetilde{M}^{(N-1)r} \widetilde{I}^r \subseteq x_0^{(N+1)r-j} \widetilde{M}^{Nr} \widetilde{I}^{j-Nr},$$

where the last inclusion comes from the fact that $j \leq (N+1)r$, hence $j - Nr \leq r$, and so using Lemma 2.4 as above we can convert $r - (j - Nr)$ of the factors of \widetilde{I}^r to \widetilde{M} , giving $\widetilde{I}^r \subseteq \widetilde{M}^{r-(j-Nr)} \widetilde{I}^{j-Nr}$.

(b) Since $r(N+1) - N \geq rN - (N-1)$, we have $\widehat{I}^{(r(N+1)-N)} \subseteq \widehat{I}^{(rN-(N-1))}$. Thus it is enough to show $\widehat{I}^{(rN-(N-1))} \subseteq \widehat{I}^r$. As in (a), for $m = rN - N + 1$, we have

$$\begin{aligned} \widehat{I}^{(m)} &= (x_0^m) + x_0^{m-1} \widetilde{I} + \cdots + x_0 \widetilde{I}^{(m-1)} + \widetilde{I}^{(m)} \\ &= x_0^r ((x_0^{m-r}) + \cdots + \widetilde{I}^{(m-r)}) + x_0^{r-1} \widetilde{I}^{(m-r+1)} + \cdots + \widetilde{I}^{(m)} \\ &= x_0^r ((x_0^{(r-1)(N-1)}) + \cdots + \widetilde{I}^{((r-1)(N-1))}) + x_0^{r-1} \widetilde{I}^{((r-1)(N-1)+1)} + \cdots + \widetilde{I}^{(rN-N+1)} \end{aligned}$$

and $\widehat{I}^r = (x_0)^r + x_0^{r-1} \widetilde{I} + \cdots + x_0 \widetilde{I}^{r-1} + \widetilde{I}^r$. Thus it suffices to show that $x_0^{r-j} \widetilde{I}^{((r-1)(N-1)+j)} \subseteq x_0^{r-j} \widetilde{I}^j$ for $1 \leq j \leq r$. But $x_0^{r-j} \widetilde{I}^{((r-1)(N-1)+j)} \subseteq x_0^{r-j} \widetilde{I}^j$ holds if $\widetilde{I}^{((r-1)(N-1)+j)} \subseteq \widetilde{I}^j$ does, which holds if $I^{((r-1)(N-1)+j)} \subseteq I^j$ does. Thus we are done, since the latter does hold: $I^{(Nj-(N-1))} \subseteq I^j$ holds by assumption, and $I^{((r-1)(N-1)+j)} \subseteq I^{(Nj-(N-1))}$ holds since $(r-1)(N-1) + j \geq Nj - (N-1)$ for $1 \leq j \leq r$.

(c) The proof for this part follows the same outline as was used for part (a). We again have $\widehat{I}^{(m)} = (x_0^m) + x_0^{m-1} \widetilde{I} + \cdots + x_0 \widetilde{I}^{(m-1)} + \widetilde{I}^{(m)}$ for any $m \geq 1$. Thus $\widehat{M}^{(r-1)N} \widehat{I}^r$ equals

$$((x_0)^{(r-1)N} + x_0^{(r-1)N-1} \widetilde{M} + \cdots + x_0 \widetilde{M}^{(r-1)N-1} + \widetilde{M}^{(r-1)N})((x_0)^r + x_0^{r-1} \widetilde{I} + \cdots + x_0 \widetilde{I}^{r-1} + \widetilde{I}^r).$$

Multiplying this out and combining like terms gives

$$\begin{aligned} &(x_0)^{rN-N+r} + \\ &(x_0)^{rN-N+r-1} (\widetilde{M} + \widetilde{I}) + \\ &(x_0)^{rN-N+r-2} (\widetilde{M}^2 + \widetilde{M}\widetilde{I} + \widetilde{I}^2) + \\ &\cdots + \\ &(x_0)^{rN-N+(r-r)} (\widetilde{M}^r + \widetilde{M}^{r-1} \widetilde{I} + \cdots + \widetilde{I}^r) + \\ &(x_0)^{rN-N-1} (\widetilde{M}^{r+1} + \widetilde{M}^r \widetilde{I} + \cdots + \widetilde{M} \widetilde{I}^r) + \\ &\cdots + \\ &(x_0)^{(rN-N)-(rN-N-r)} (\widetilde{M}^{rN-N} + \widetilde{M}^{rN-N-1} \widetilde{I} + \cdots + \widetilde{M}^{rN-N-r} \widetilde{I}^r) + \\ &(x_0)^{r-1} (\widetilde{M}^{rN-N} \widetilde{I} + \widetilde{M}^{rN-N-1} \widetilde{I}^2 + \cdots + \widetilde{M}^{rN-N+1-r} \widetilde{I}^r) + \\ &\cdots + \\ &(x_0) (\widetilde{M}^{rN-N} \widetilde{I}^{r-1} + \widetilde{M}^{rN-N-1} \widetilde{I}^r) + \\ &\widetilde{M}^{rN-N} \widetilde{I}^r. \end{aligned}$$

As in the proof of part (a), the first term in each row of the displayed sum of ideals above contains the other terms in that row. This gives

$$\begin{aligned} \widehat{M}^{(r-1)N} \widehat{I}^r &= (x_0)^{(r-1)N+r} + (x_0)^{(r-1)N+r-1} \widetilde{M} + \cdots + (x_0)^r \widetilde{M}^{(r-1)N} + (x_0)^{r-1} \widetilde{M}^{(r-1)N} \widetilde{I} + \\ &\cdots + x_0 \widetilde{M}^{N(r-1)} \widetilde{I}^{r-1} + \widetilde{M}^{N(r-1)} \widetilde{I}^r. \end{aligned}$$

Now

$$\widehat{I}^{(r(N+1)-N)} = (x_0^{r(N+1)-N}) + x_0^{r(N+1)-N-1}\widetilde{I} + \dots + x_0\widetilde{I}^{(r(N+1)-N-1)} + \widetilde{I}^{(r(N+1)-N)},$$

and so to show $\widehat{I}^{(r(N+1)-N)} \subseteq \widehat{M}^{(r-1)N}\widehat{I}^r$ it suffices to show that $x_0^{r(N+1)-N-j}\widetilde{I}^{(j)} \subseteq x_0^{r(N+1)-N-j}\widetilde{M}^j$ for $j = 0, \dots, rN - N$, and that $x_0^{r(N+1)-N-j}\widetilde{I}^{(j)} \subseteq x_0^{r(N+1)-N-j}\widetilde{M}^{(r-1)N}\widetilde{I}^{j-rN+N}$ for $j = rN - N + 1, \dots, rN + r - N$.

We know from the proof of part (a) that $\widetilde{I}^{(t)} \subseteq \widetilde{M}^t$ for all $t > 0$. It follows that $x_0^{r(N+1)-N-j}\widetilde{I}^{(j)} \subseteq x_0^{r(N+1)-N-j}\widetilde{M}^j$ for $j = 0, \dots, rN - N$. So, it remains to show

$$x_0^{r(N+1)-N-j}\widetilde{I}^{(j)} \subseteq x_0^{r(N+1)-N-j}\widetilde{M}^{(r-1)N}\widetilde{I}^{j-rN+N}$$

for $j = rN - N + 1, \dots, rN + r - N$. To this end, note that, since $j \geq rN - N + 1$, we know that $j - (rN - N + 1) \geq 0$ and so Lemma 2.4 gives $\widetilde{I}^{(j)} \subseteq M^{j-(rN-N+1)}\widetilde{I}^{(rN-N+1)}$ and hence

$$x_0^{r(N+1)-N-j}\widetilde{I}^{(j)} \subseteq x_0^{r(N+1)-N-j}\widetilde{M}^{j-(rN-N+1)}\widetilde{I}^{(rN-N+1)}.$$

By assumption, $\widetilde{I}^{(rN-N+1)} \subseteq M^{(r-1)(N-1)}I^r$ which implies that $\widetilde{I}^{(rN-N+1)} \subseteq \widetilde{M}^{(r-1)(N-1)}\widetilde{I}^r$. Hence we have

$$x_0^{r(N+1)-N-j}\widetilde{I}^{(j)} \subseteq x_0^{r(N+1)-N-j}\widetilde{M}^{j-(rN-N+1)}\widetilde{M}^{(r-1)(N-1)}\widetilde{I}^r \subseteq x_0^{r(N+1)-N-j}\widetilde{M}^{rN-N}\widetilde{I}^{j-rN+N},$$

where the last inclusion comes from converting $r - (j - rN + N)$ of the factors of \widetilde{I}^r to \widetilde{M} , resulting in $\widetilde{I}^r \subseteq \widetilde{M}^{r-(j-rN+N)}\widetilde{I}^{j-rN+N}$. (This conversion can be done via Lemma 2.4 since $j \leq rN + r - N$.) \square

As a corollary of the preceding result, we have the following theorem.

Theorem 7.2. *Assume K has characteristic 0. Let $Z = P_1 + \dots + P_n$ be a star configuration of points in \mathbf{P}^N , where we regard \mathbf{P}^N as a linear subspace of a larger dimensional projective space \mathbf{P}^d . Let $I = I(Z_{\mathbf{P}^d})$ and let M be the irrelevant ideal for $K[\mathbf{P}^d]$. Then for all $r \geq 1$ we have:*

- (a) $I^{(rd)} \subseteq M^{r(d-1)}I^r$ and
- (b) $I^{(rd-(d-1))} \subseteq M^{(r-1)(d-1)}I^r$.

Proof. (a) As noted in Section 6, Conjecture 3.1 holds for star configurations of points, hence the result follows by Proposition 7.1(a).

(b) Also as noted in Section 6, Conjecture 3.4 holds for star configurations of points, hence the result follows by Proposition 7.1(c). \square

As another corollary we have the following result. Part (a) was proved in [Du] using different methods.

Theorem 7.3. *Assume K has characteristic 0. Let $Z = P_1 + \dots + P_n$ for $n \leq d + 1$ general points of \mathbf{P}^d . Let $I = I(Z)$ and let M be the irrelevant ideal for $K[\mathbf{P}^d]$. Then for all $r \geq 1$ we have:*

- (a) $I^{(rd)} \subseteq M^{r(d-1)}I^r$ and
- (b) $I^{(rd-(d-1))} \subseteq M^{(r-1)(d-1)}I^r$.

Proof. Just note that Z is a star configuration of points in \mathbf{P}^{n-1} and apply Theorem 7.2. \square

8. GENERAL POINTS IN THE PLANE

Our focus here is for general points in the plane. (Apart from [Du], which shows that Conjecture 3.1 holds for finite sets of general points in \mathbf{P}^3 , little so far is known for general points in \mathbf{P}^N for $N > 2$.)

Let I be the radical ideal of n general points in \mathbf{P}^2 and let $M \subset K[\mathbf{P}^2]$ be the maximal proper homogeneous ideal (i.e., the irrelevant ideal). If n is 1, 2 or 4, then Conjectures 3.1 through 3.9 hold since the points comprise a complete intersection. If $n = 3$ or 5, Conjectures 3.1 through 3.9 hold since the points comprise a star configuration (in case $n = 3$) or lie on a smooth conic (if $n = 5$),

although for Conjectures 3.6, 3.7, 3.8 and 3.9 our proof above for both $n = 3$ and 5 assumes that the characteristic is 0.

So now assume that $n \geq 6$ and $N = 2$.

Conjecture 3.1, that $I^{(rN)} \subseteq M^{r(N-1)}I^r$, holds by [HaHu, Corollary 3.10].

Conjecture 3.2, that $I^{(rN-(N-1))} \subseteq I^r$, holds by [BH2, Remark 4.3], since $(rN - (N - 1))/r = 2 - (1/r) \geq 3/2$ except in the trivial case that $r = 1$.

Conjecture 3.3, that $I^{(m)} \subseteq I^r$ if $m/r \geq 2\alpha(I)/(\alpha(I) + 1)$, holds (see the discussion after Conjecture 4.1.4 of [HaHu]).

Conjecture 3.4, that $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)}I^r$, holds by [HaHu, Corollary 4.1.13] but in some cases the proof assumes $\text{char}(K) = 0$.

Conjecture 3.5, that $\alpha(I^{(rN-(N-1))}) \geq r\alpha(I) + (r-1)(N-1)$, follows from Conjecture 3.4.

Conjecture 3.6, that $\frac{\alpha(I^{(m)})+N-1}{m+N-1} \leq \frac{\alpha(I^{(r)})}{r}$, and Conjecture 3.7, that $I^{(t(m+N-1))} \subseteq M^t(I^{(m)})^t$, follow from Conjecture 3.8, that $I^{(t(m+N-1))} \subseteq M^{t(N-1)}(I^{(m)})^t$. We now consider some special cases of the latter, and of Conjecture 3.9, that $I^{(t(m+N-1)-N+1)} \subseteq (I^{(m)})^t$ and $I^{(t(m+N-1)-N+1)} \subseteq M^{(t-1)(N-1)}(I^{(m)})^t$, for $N = 2$.

Proposition 8.1. *Let $m, t \geq 1$ and let I be the radical ideal of n general points in \mathbf{P}^2 where $n = s^2$ for $s \geq 3$. Then $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$, $I^{(t(m+1)-1)} \subseteq (I^{(m)})^t$ and $I^{(t(m+1)-1)} \subseteq M^{t-1}(I^{(m)})^t$.*

Proof. Let $n = 9$. Then $\alpha(I^{(r)}) = 3r$ and $\text{reg}(I^{(r)}) = 3r + 1$ for $r \geq 1$ [H1]. Thus $\alpha(I^{(t(m+1))}) = 3t(m+1) \geq 3tm + 2t = t\text{reg}(I^{(m)}) + t$ holds so by Lemma 2.6 we have $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$, which verifies Conjecture 3.8. Also, $\alpha(I^{(t(m+1)-1)}) = 3tm + 3t - 3$, and $\text{reg}((I^{(m)})^t) \leq t(3m + 1)$ by [GGP]. Consider $I^{(t(m+1)-1)} \subseteq (I^{(m)})^t$. If $t = 1$, we have $I^{(t(m+1)-1)} = I^{(m)} = (I^{(m)})^t$, so assume $t > 1$. For $i < 3tm + 3t - 3$, we have $(I^{(t(m+1)-1)})_i = 0 \subseteq ((I^{(m)})^t)_i$, while for $i \geq 3tm + 3t - 3$, we have $i \geq 3tm + 3t - 3 \geq t(3m + 1) \geq \text{reg}((I^{(m)})^t)$, so $(I^{(t(m+1)-1)})_i \subseteq (I^{(m)})_i = ((I^{(m)})^t)_i$. Thus $I^{(t(m+1)-1)} \subseteq (I^{(m)})^t$ holds and $I^{(t(m+1)-1)} \subseteq M^{t-1}(I^{(m)})^t$ now follows from Lemma 2.6(b).

For $s > 3$, the proof is similar except we use $\alpha(I^{(r)}) > rs$ [N] and $\text{reg}(I^{(r)}) \leq s(r + \frac{1}{2})$ [HHF, Lemma 2.5]. For example, $t(m+1)s \geq ts(m + \frac{1}{2}) + t$ for $s \geq 2$, so we have $\alpha(I^{(t(m+1))}) > t(m+1)s \geq ts(m + \frac{1}{2}) + t \geq t\text{reg}(I^{(m)}) + t$. Thus $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$ holds by Lemma 2.6. Similarly, we conclude $I^{(t(m+1)-1)} \subseteq (I^{(m)})^t$ and $I^{(t(m+1)-1)} \subseteq M^{t-1}(I^{(m)})^t$. \square

Remark 8.2. For $n > 9$ if we assume the well-known SHGH Conjecture (that $\dim_K(I^{(m)})_t = \max(0, \binom{t+2}{2} - n\binom{m+1}{2})$), then we get $\alpha(I^{(t(m+1))}) \geq t(m+1)\sqrt{n}$ and $\text{reg}(I^{(m)}) < (m + (1/2))\sqrt{n} + 1$. This implies $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$ holds if $t((m + (1/2))\sqrt{n} + 1) + t \leq t(m+1)\sqrt{n}$, and this is true for $n \geq 16$.

For $n \geq 10$ generic points, using known estimates for $\alpha(I^{(m)})$ and $\text{reg}(I^{(m)})$ we can verify Conjecture 3.8 for all t for $1 \leq m \leq \sqrt{n+1} - 1$.

Proposition 8.3. *Let I be the radical ideal of n generic points in \mathbf{P}^2 where $n \geq 10$. Then $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$ for all t for $1 \leq m \leq \sqrt{n+1} - 1$.*

Proof. By Proposition 8.1 we may as well assume that n is not a square. The Seshadri constant $\varepsilon(n)$ for n generic points is at least $\varepsilon(n) \geq 1/\sqrt{n+1}$ by [H3, Proposition I.2(c)] and [H3, Proposition I.3]. This means that $\alpha(I^{(t(m+1))}) \geq t(m+1)n/\sqrt{n+1}$. By [H3, Corollary V.2.2(b)], for $n > 9$ not a square we have $\text{reg}(I^{(m)}) \leq \frac{m+1}{\varepsilon(n)} - 2 \leq (m+1)\sqrt{n+1} - 2$. But $t(m+1)n/\sqrt{n+1} \geq t((m+1)\sqrt{n+1} - 2) + t$

holds for $1 \leq m \leq \sqrt{n+1} - 1$. Thus by Lemma 2.6 we have $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$ for $1 \leq m \leq \sqrt{n+1} - 1$. \square

9. ADDITIONAL RESULTS

It's possible to give a partial verification of Conjecture 3.6.

Proposition 9.1. *Let I be the radical ideal of a finite set of points in \mathbf{P}^N . Assume $\text{char}(K) = 0$.*

(1) *If $r \leq N$, then*

$$\frac{\alpha(I) + N - 1}{N} \leq \frac{\alpha(I^{(r)})}{r}.$$

(2) *If $m \leq r \leq N$, then*

$$\frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \leq \frac{\alpha(I^{(r)})}{r}.$$

(3) *If either $\alpha(I^{(m)}) = m$, or $m \leq r$ and $r - m < N$, then*

$$\frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \leq \frac{\alpha(I^{(r)})}{r}.$$

Proof. Note that (1) is a special case of (2). For (2), assume $(\alpha(I^{(m)}) + N - 1)/(m + N - 1) > \alpha(I^{(r)})/r$. We know $\alpha(I^{(r)}) \geq \alpha(I^{(m)}) + r - m$, by repeated application of Lemma 2.4. Thus we have $(\alpha(I^{(m)}) + N - 1)/(m + N - 1) > (\alpha(I^{(m)}) + r - m)/r$. This simplifies to $(r - m)(\alpha(I^{(m)}) - m) > (N - 1)(\alpha(I^{(m)}) - m)$, which is impossible.

(3) If $\alpha(I^{(m)}) = m$, then we must show $1 \leq \alpha(I^{(r)})/r$, but this holds since $\alpha(I^{(r)}) \geq r$. If $m \leq r$ and $r - m < N$, then the proof is the same as the proof of (2), since $\alpha(I^{(m)}) - m \geq 0$. \square

Conjectures 3.7 and 3.8 are much stronger than what is currently known. The best general result known along these lines for the radical ideal I of a finite set of points in \mathbf{P}^N is that

$$I^{(t(N+m-1)+1)} \subseteq M(I^{(m)})^t$$

for all $t, m \geq 1$; this follows from [TY, Theorem 0.1(1)]. Here we strengthen this result if $\text{char}(K) = 0$.

Proposition 9.2. *Let I be the radical ideal of a finite set of points in \mathbf{P}^N . Assume $\text{char}(K) = 0$. Then*

$$I^{(t(N+m-1)+s)} \subseteq M^s(I^{(m)})^t$$

for all $s, t, m \geq 1$.

Proof. By Lemma 2.4 we have $I^{(Nt+(m-1)t+s)} \subseteq M I^{(Nt+(m-1)t+s-1)}$. Repeating and using Theorem 2.2 that $I^{(Nt+(m-1)t)} \subseteq (I^{(m)})^t$, gives $I^{(Nt+(m-1)t+s)} \subseteq M^s I^{(Nt+(m-1)t)} \subseteq M^s (I^{(m)})^t$. \square

Finally, Conjecture 3.5 has been verified in separate joint work by the second author for various special configurations of points in \mathbf{P}^2 in characteristic 0 (the configurations in question consist of certain unions of sets of collinear points called line count configurations); see [CH].

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